CHAPTER 1

A toolbox

Objectives

- To revise the properties of sine, cosine and tangent
- To revise methods for solving right-angled triangles
- To revise the sine rule and cosine rule
- To revise basic triangle, parallel line and circle geometry
- To revise arithmetic and geometric sequences
- To revise arithmetic and geometric series
- To revise infinite geometric series
- To revise cartesian equations for circles
- To sketch graphs of ellipses from the general cartesian relation
  \[
  \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
  \]
- To sketch graphs of hyperbolas from the general cartesian relation
  \[
  \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1
  \]
- To consider asymptotic behaviour of hyperbolas
- To work with parametric equations for circles, ellipses and hyperbolas

The first six sections of this chapter revise areas for which knowledge is required in this course, and which are referred to in the Specialist Mathematics Study Design.

The final section introduces cartesian and parametric equations for ellipses and hyperbolas.

1.1 Circular functions

Defining sine, cosine and tangent

The unit circle is a circle of radius one with centre at the origin.

It is the graph of the relation \(x^2 + y^2 = 1\).
Sine and cosine may be defined for any angle through the unit circle.

For the angle of $\theta^\circ$, a point $P$ on the unit circle is defined as illustrated opposite. The angle is measured in an anticlockwise direction from the positive direction of the $x$-axis.

$\cos(\theta^\circ)$ is defined as the $x$-coordinate of the point $P$ and $\sin(\theta^\circ)$ is defined as the $y$-coordinate of $P$. A calculator gives approximate values for these coordinates where the angle is given.

$\sin 30^\circ = 0.5$ (exact value)
$\cos 30^\circ = \frac{\sqrt{3}}{2} \approx 0.8660$

$\sin 135^\circ = \frac{1}{\sqrt{2}} \approx 0.7071$
$\cos 135^\circ = -\frac{1}{\sqrt{2}} \approx -0.7071$
$\sin 100^\circ \approx 0.9848$

$\tan(\theta^\circ)$ is defined by $\tan(\theta^\circ) = \frac{\sin(\theta^\circ)}{\cos(\theta^\circ)}$. The value of $\tan(\theta^\circ)$ can be illustrated geometrically through the unit circle.

For a right-angled triangle $OBC$, a similar triangle $OB'C'$ can be constructed that lies in the unit circle.

By the definition, $OC' = \cos(\theta^\circ)$ and $CB' = \sin(\theta^\circ)$.
The scale factor is the length $OB$.
Hence $BC = OB \sin(\theta^\circ)$ and $OC = OB \cos(\theta^\circ)$.
This implies

$$\frac{BC}{OB} = \sin(\theta^\circ) \quad \text{and} \quad \frac{OC}{OB} = \cos(\theta^\circ)$$
This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle $\theta^\circ$ is as shown.

$$\sin \theta^\circ = \frac{\text{opp}}{\text{hyp}} \quad \text{cos} \theta^\circ = \frac{\text{adj}}{\text{hyp}} \quad \text{tan} \theta^\circ = \frac{\text{opp}}{\text{adj}}$$

### Definition of a radian

In moving around the circle a distance of 1 unit from $A$ to $P$ the angle $POA$ is defined. The measure of this angle is 1 radian.

One radian (written $1^c$) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

**Note:** Angles formed by moving anticlockwise around the circumference of the unit circle are defined as positive. Those formed by moving in a clockwise direction are said to be negative.

### Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$

$$\therefore \quad 2\pi^c = 360^\circ$$

$$\therefore \quad \pi^c = 180^\circ$$

$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Henceforth the $^c$ may be omitted. Any angle is assumed to be measured in radians unless otherwise indicated.

The following table displays the conversions of some special angles from degrees to radians.

<table>
<thead>
<tr>
<th>Angles in degrees</th>
<th>0</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>90</th>
<th>180</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angles in radians</td>
<td>$0$</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$2\pi$</td>
</tr>
</tbody>
</table>
Some values for the trigonometric functions are given in the following table.

<table>
<thead>
<tr>
<th>$x$ in radians</th>
<th>$\sin x$</th>
<th>$\cos x$</th>
<th>$\tan x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
<td>0</td>
<td>undefined</td>
</tr>
</tbody>
</table>

The graphs of sine and cosine

As $\sin x = \sin(x + 2\pi n)$, $n \in \mathbb{Z}$, the function is periodic and the period is $2\pi$.

A sketch of the graph of $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \sin x$ is shown opposite. The amplitude is 1.

A sketch of the graph of $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \cos x$ is shown opposite. The period of the function is $2\pi$. The amplitude is 1.

For $y = a \cos(nx)$ and $y = a \sin(nx)$, $a > 0$, $n > 0$

Period = $\frac{2\pi}{n}$, amplitude = $a$, range = $[-a, a]$

Symmetry properties for sine and cosine

From the graph of the functions or from the unit circle definitions, the following results may be obtained.

- $\sin(\pi - \theta) = \sin \theta$
- $\cos(\pi - \theta) = -\cos \theta$
- $\sin(2\pi - \theta) = -\sin \theta$
- $\cos(2\pi - \theta) = \cos \theta$
- $\sin(\pi + \theta) = -\sin \theta$
- $\cos(\pi + \theta) = -\cos \theta$
- $\sin(-\theta) = -\sin \theta$
- $\cos(-\theta) = \cos \theta$
- $\sin(\theta + 2\pi n) = \sin \theta$
- $\cos(\theta + 2\pi n) = \cos \theta$ for $n \in \mathbb{Z}$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$
- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$
**Example 1**

a  Change $135^\circ$ into radians.

b  Change $1.5^\circ$ into degrees, correct to two decimal places.

c  Find the exact value of:

i  $\sin \pi^c$

ii  $\cos \left( \frac{7\pi}{4} \right)^c$

**Solution**

a  $135^\circ = \left( \frac{135 \times \pi}{180} \right)^c = \left( \frac{3\pi}{4} \right)^c$

Note that angles in radians which are expressed in terms of $\pi$ are left in that form.

b  $1.5^c = \left( \frac{1.5 \times 180}{\pi} \right)^\circ = 85.94^\circ$, correct to two decimal places.

c  i  $\sin \pi^c = 0$

ii  $\cos \left( \frac{7\pi}{4} \right)^c = \cos \left( \frac{7\pi}{4} - 2\pi \right)^c = \cos \left( \frac{-\pi}{4} \right)^c = \cos \frac{\pi^c}{4} = \frac{\sqrt{2}}{2}$

**Example 2**

Find the exact value of:

a  $\sin 150^\circ$

b  $\cos(-585^\circ)$

**Solution**

a  $\sin 150^\circ = \sin(180^\circ - 150^\circ) = \sin 30^\circ = 0.5$

b  $\cos(-585^\circ) = \cos 585^\circ = \cos(585^\circ - 360^\circ) = \cos(225^\circ) = -\cos 45^\circ = -\frac{\sqrt{2}}{2}$

**Example 3**

Find the exact value of:

a  $\sin \left( \frac{11\pi}{6} \right)$

b  $\cos \left( \frac{-45\pi}{6} \right)$

**Solution**

a  $\sin \left( \frac{11\pi}{6} \right) = \sin \left( 2\pi - \frac{11\pi}{6} \right) = -\sin \frac{\pi}{6} = -\frac{1}{2}$

b  $\cos \left( \frac{-45\pi}{6} \right) = \cos \left( -\frac{7\frac{1}{2} \times \pi}{2} \right) = \cos \left( \frac{-\pi}{2} \right) = 0$
The Pythagorean identity

For any value of \( \theta \)

\[
\cos^2 \theta + \sin^2 \theta = 1
\]

Example 4

If \( \sin x^\circ = 0.3, 0 < x < 90 \), find:

\( a \) \( \cos x^\circ \) \( b \) \( \tan x^\circ \)

Solution

\( a \) \( \sin^2 x^\circ + \cos^2 x^\circ = 1 \)

\( 0.09 + \cos^2 x^\circ = 1 \)

\( \therefore \cos^2 x^\circ = 0.91 \)

\( \cos x^\circ = \pm \sqrt{0.91} \)

as \( 0 < x < 90 \), \( \cos x^\circ = \sqrt{0.91} = \sqrt{91}/10 \)

\( b \) \( \tan x^\circ = \frac{\sin x}{\cos x} = \frac{0.3}{\sqrt{0.91}} = \frac{3}{\sqrt{91}} = 3\sqrt{91}/91 \)

Solution of equations

If a trigonometric equation has a solution, then it will have a corresponding solution in each ‘cycle’ of its domain. Such equations are solved by using the symmetry of the graphs to obtain solutions within one ‘cycle’ of the function. Other solutions may be obtained by adding multiples of the period to these solutions.

Example 5

The graph of \( y = f(x) \) where \( f: R \to R, f(x) = \sin x, x \in [0, 2\pi] \) is shown.

Find the other \( x \) value which has the same \( y \) value as each of the pronumerals marked.

Solution

For \( x = a \), the value is \( \pi - a \).

For \( x = b \), the other value is \( \pi - b \).

For \( x = c \), the other value is \( 2\pi - (c - \pi) = 3\pi - c \).

For \( x = d \), the other value is \( \pi + (2\pi - d) = 3\pi - d \).

Example 6

Solve the equation \( \sin \left( 2x + \frac{\pi}{3} \right) = \frac{1}{2} \) for \( x \in [0, 2\pi] \).
Solution

Let \( \theta = 2x + \frac{\pi}{3} \)

Note

\[ 0 \leq x \leq 2\pi \]
\[ \Leftrightarrow 0 \leq 2x \leq 4\pi \]
\[ \Leftrightarrow \frac{\pi}{3} \leq 2x + \frac{\pi}{3} \leq \frac{13\pi}{3} \]
\[ \Leftrightarrow \frac{\pi}{3} \leq \theta \leq \frac{13\pi}{3} \]

Therefore solving the equation \( \sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \) for \( x \in [0, 2\pi] \) is achieved by first solving the equation \( \sin(\theta) = \frac{1}{2} \) for \( \frac{\pi}{3} \leq \theta \leq \frac{13\pi}{3} \)

Consider \( \sin \theta = \frac{1}{2} \)

\[ \Leftrightarrow \theta = \frac{\pi}{6} \text{ or } 5\frac{\pi}{6} \text{ or } 2\pi + \frac{\pi}{6} \text{ or } 2\pi + 5\frac{\pi}{6} \text{ or } 4\pi + \frac{\pi}{6} \text{ or } 4\pi + 5\frac{\pi}{6} \text{ or } \ldots \]

The solutions \( \frac{\pi}{6} \) and \( \frac{29\pi}{6} \) are not required as they lie outside the restricted domain for \( \theta \).

\[ \therefore \text{For } \frac{\pi}{3} \leq \theta \leq \frac{13\pi}{3} \]

\[ \theta = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6} \]

\[ \therefore 2x + \frac{2\pi}{6} = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6} \]

\[ \therefore 2x = \frac{3\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{15\pi}{6} \text{ or } \frac{23\pi}{6} \]

\[ \therefore x = \frac{\pi}{4} \text{ or } \frac{11\pi}{12} \text{ or } \frac{5\pi}{4} \text{ or } \frac{23\pi}{12} \]

Using a graphics calculator

A graphics calculator can be used to find a numerical solution to the equation \( \sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \) by plotting the graphs of \( y = \sin\left(2x + \frac{\pi}{3}\right) \) and \( y = \frac{1}{2} \) and considering the points of intersection.

It can be seen from the graph for \([0, 2\pi]\) that there are four solutions. The \( x \) coordinates of the points of intersection are found through using 5:intersect from the CALC menu.

It is sometimes possible to find the exact value by calculating \( X/\pi \). The corresponding exact value is \( \frac{11\pi}{12} \).
Using a CAS calculator

A CAS calculator can be used to solve this equation. The syntax is `solve(sin(2x + π/3) = 1/2, x | 0 ≤ x and x ≤ 2π).

The result is shown.

Sketch graphs

The graphs of functions defined by rules of the form \( f(x) = a \sin(nx + \varepsilon) + b \) and \( f(x) = a \cos(nx + \varepsilon) + b \) can be obtained from the graphs of \( \sin x \) and \( \cos x \) by transformations.

Example 7

Sketch the graph of \( h: [0, 2\pi] \to \mathbb{R}, h(x) = 3 \cos \left(2x + \frac{\pi}{3}\right) + 1. \)

Solution

\[
h(x) = 3 \cos \left(2 \left(x + \frac{\pi}{6}\right)\right) + 1
\]

The transformations from the graph of \( y = \cos x \) are:
- a dilation from the \( y \)-axis of factor \( \frac{1}{2} \)
- a dilation from the \( x \)-axis of factor \( 3 \)
- a translation of \( \frac{\pi}{6} \) in the negative direction of the \( x \) axis
- a translation of \( 1 \) in the positive direction of the \( y \) axis

The graph with the dilations applied is as shown below.

The translation \( \frac{\pi}{6} \) in the negative direction of the \( x \) axis is then applied.
The final translation is applied and the graph is given for the required domain.

The x-axis intercepts are found by solving the equation.

\[ 3 \cos \left(2x + \frac{\pi}{3}\right) + 1 = 0 \]

i.e.

\[ \cos \left(2x + \frac{\pi}{3}\right) = -\frac{1}{3} \]

The graph of tan

A sketch of the graph of \( f: R \setminus \{(2n + 1)\frac{\pi}{2}; n \in Z\} \rightarrow R, f(\theta) = \tan \theta \) is shown below.

**Note:** \( \theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \) and \( \frac{5\pi}{2} \) are asymptotes.

**Observations from the graph**

- The graph repeats itself every \( \pi \) units, i.e. the period of \( \tan \) is \( \pi \).
- Range of \( \tan \) is \( R \).
- The vertical asymptotes have equation \( \theta = (2k + 1)\frac{\pi}{2} \) where \( k \in Z \).
Using a graphics calculator

If the normal procedure is followed and \( y = \tan x \) is entered into the \( Y = \) menu and then plotted by pressing [ZOOM] 7, we get the graph as shown here.

While it might appear that the calculator has drawn in asymptotes, it hasn’t. Instead, it has simply joined up the last two points plotted either side of the point where the asymptote lies and joined them up.

Note that, since \( \tan x = \frac{\sin x}{\cos x} \), the domain of \( \tan \) must exclude all real numbers where \( \cos x \) is zero, i.e. all odd integral multiples of \( \frac{\pi}{2} \). The graph shows vertical asymptotes at \( x = \frac{(2n + 1)\pi}{2} \). This function has a period of \( \pi \) as \( \tan x = \tan(x + n\pi), \ n \in \mathbb{Z} \). The concept of amplitude is not applicable here.

Symmetry properties for \( \tan \)

From the definition of \( \tan \), the following results are obtained:

\[
\tan(\pi - \theta) = -\tan \theta \quad \tan(\pi + \theta) = \tan \theta \quad \tan(2\pi - \theta) = -\tan \theta \quad \tan(-\theta) = -\tan \theta
\]

Example 8

Find the exact values of:

a \( \tan (330^\circ) \)  
b \( \tan \left(\frac{4\pi}{3}\right) \)

Solution

a \( \tan (330^\circ) = \tan(360 - 30)^\circ = -\tan(30)^\circ = -\frac{\sqrt{3}}{3} \)

b \( \tan \left(\frac{4\pi}{3}\right) = \tan \left(\pi + \frac{\pi}{3}\right) = \tan \left(\frac{\pi}{3}\right) = \sqrt{3} \)

Solutions of equations involving \( \tan \)

The procedure here is similar to that used for solving equations involving \( \sin \) and \( \cos \), except that only one solution needs to be selected then all other solutions are \( \pi \) or \( 180^\circ \) apart.

Example 9

Solve the equations:

a \( \tan x = -1 \) for \( x \in [0, 4\pi] \)  
b \( \tan(2x - \pi) = \sqrt{3} \) for \( x \in [-\pi, \pi] \)

Solution

a \( \tan x = -1 \)

Now \( \tan \frac{3\pi}{4} = -1 \).
Therefore \( x = \frac{3\pi}{4} \) or \( \frac{3\pi}{4} + \pi \) or \( \frac{3\pi}{4} + 2\pi \) or \( \frac{3\pi}{4} + 3\pi \).

Hence \( x = \frac{3\pi}{4} \) or \( \frac{7\pi}{4} \) or \( \frac{11\pi}{4} \) or \( \frac{15\pi}{4} \).

b \( \tan(2x - \pi) = \sqrt{3} \).

Therefore \(-2\pi \leq 2x \leq 2\pi\) and thus \(-3\pi \leq 2x - \pi \leq \pi\) and \(-3\pi \leq \theta \leq \pi\).

In order to solve \( \tan(2x - \pi) = \sqrt{3} \) first solve \( \tan \theta = \sqrt{3} \).

\[
\begin{align*}
\theta &= \frac{\pi}{3} \quad \text{or} \quad \frac{\pi}{3} - \pi \quad \text{or} \quad \frac{\pi}{3} - 2\pi \quad \text{or} \quad \frac{\pi}{3} - 3\pi \\
\therefore \quad \theta &= \frac{\pi}{3} \quad \text{or} \quad -\frac{2\pi}{3} \quad \text{or} \quad -\frac{5\pi}{3} \quad \text{or} \quad -\frac{8\pi}{3} \\
\end{align*}
\]

and as \( \theta = 2x - \pi \)

\[
\begin{align*}
2x - \pi &= \frac{\pi}{3} \quad \text{or} \quad -\frac{2\pi}{3} \quad \text{or} \quad -\frac{5\pi}{3} \quad \text{or} \quad -\frac{8\pi}{3} \\
\end{align*}
\]

Therefore \( 2x = \frac{4\pi}{3} \) or \( \frac{\pi}{3} \) or \( -\frac{2\pi}{3} \) or \( -\frac{5\pi}{3} \).

And \( x = \frac{2\pi}{3} \) or \( \frac{\pi}{6} \) or \( -\frac{\pi}{3} \) or \( -\frac{5\pi}{6} \).

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**Exercise 1A**

1 a Change the following angles from degrees to exact values in radians:

\( i \) 720° \( ii \) 540° \( iii \) -450° \( iv \) 15° \( v \) -10° \( vi \) -315°

b Change the following angles from radians to degrees:

\( i \) \( \frac{5\pi}{4} \) \( ii \) \( -\frac{2\pi}{3} \) \( iii \) \( \frac{7\pi}{12} \) \( iv \) \( -\frac{11\pi}{6} \) \( v \) \( \frac{13\pi}{9} \) \( vi \) \( -\frac{11\pi}{12} \)

2 Perform the correct conversion on each of the following, giving the answer correct to two decimal places.

a Convert from degrees to radians:

\( i \) 7° \( ii \) -100° \( iii \) -25° \( iv \) 51° \( v \) 206° \( vi \) -410°

b Convert from radians to degrees:

\( i \) 1.7° \( ii \) -0.87° \( iii \) 2.8° \( iv \) 0.1° \( v \) -3° \( vi \) -8.9°

3 Find the exact value of each of the following:

a \( \sin \left( \frac{2\pi}{3} \right) \) \( b \cos \left( \frac{3\pi}{4} \right) \) \( c \cos \left( -\frac{\pi}{3} \right) \)

d \( \cos \left( \frac{5\pi}{4} \right) \) \( e \cos \left( \frac{9\pi}{4} \right) \) \( f \sin \left( \frac{11\pi}{3} \right) \)

\( g \cos \left( \frac{31\pi}{6} \right) \) \( h \cos \left( \frac{29\pi}{6} \right) \) \( i \sin \left( -\frac{23\pi}{6} \right) \)

4 Find the exact value of each of the following:

a \( \sin(135°) \) \( b \cos(-300°) \) \( c \sin(480°) \)

d \( \cos(240°) \) \( e \sin(-225°) \) \( f \sin(420°) \)
5 If \( \sin(x) = 0.5 \) and \( 90 < x < 180 \), find:
   a \( \cos(x) \)  
   b \( \tan(x) \)

6 If \( \cos(x) = -0.7 \) and \( 180 < x < 270 \), find:
   a \( \sin(x) \)  
   b \( \tan(x) \)

7 If \( \sin(x) = -0.5 \) and \( \pi < x < \frac{3\pi}{2} \), find:
   a \( \cos(x) \)  
   b \( \tan(x) \)

8 If \( \sin(x) = -0.3 \) and \( \frac{3\pi}{2} < x < 2\pi \), find:
   a \( \cos(x) \)  
   b \( \tan(x) \)

9 Solve each of the following for \( x \in [0, 2\pi] \):
   a \( \sin x = -\frac{\sqrt{3}}{2} \)  
   b \( \sin (2x) = \frac{\sqrt{3}}{2} \)  
   c \( 2 \cos (2x) = -1 \)  
   d \( \sin(x + \frac{\pi}{3}) = -\frac{1}{2} \)  
   e \( 2 \cos \left(2 \left(x + \frac{\pi}{3}\right)\right) = -1 \)  
   f \( 2 \sin \left(2x + \frac{\pi}{3}\right) = -\sqrt{3} \)

10 Find the exact values of each of the following:
   a \( \tan \left(\frac{5\pi}{4}\right) \)  
   b \( \tan \left(-\frac{2\pi}{3}\right) \)  
   c \( \tan \left(-\frac{29\pi}{6}\right) \)  
   d \( \tan(240^\circ) \)

11 If \( \tan x = -\frac{1}{4} \) and \( \pi \leq x \leq \frac{3\pi}{2} \), find the exact value of:
   a \( \sin x \)  
   b \( \cos x \)  
   c \( \tan(-x) \)  
   d \( \tan(\pi - x) \)

12 If \( \tan x = -\frac{\sqrt{3}}{2} \) and \( \frac{\pi}{2} \leq x \leq \pi \), find the exact value of:
   a \( \sin x \)  
   b \( \cos x \)  
   c \( \tan(-x) \)  
   d \( \tan(x - \pi) \)

13 Solve each of the following for \( x \in [0, 2\pi] \):
   a \( \tan x = -\sqrt{3} \)  
   b \( \tan \left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \)  
   c \( 2 \tan \left(\frac{x}{2}\right) + 2 = 0 \)  
   d \( 3 \tan \left(\frac{\pi}{2} + 2x\right) = -3 \)

14 Sketch the graphs of each of the following for the stated domain:
   a \( f(x) = \sin 2x, x \in [0, 2\pi] \)  
   b \( f(x) = \cos \left(x + \frac{\pi}{3}\right), x \in \left[-\frac{\pi}{3}, \pi\right] \)  
   c \( f(x) = \cos \left(2 \left(x + \frac{\pi}{3}\right)\right), x \in [0, \pi] \)  
   d \( f(x) = 2 \sin(3x) + 1, x \in [0, \pi] \)  
   e \( f(x) = 2 \sin \left(x - \frac{\pi}{4}\right) + \sqrt{3}, x \in [0, 2\pi] \)

15 Sketch the graphs of each of the following for \( x \in [0, \pi] \), clearly labelling all intercepts with the axes and all asymptotes:
   a \( f(x) = \tan(2x) \)  
   b \( f(x) = \tan \left(x - \frac{\pi}{3}\right) \)  
   c \( f(x) = 2 \tan \left(2x + \frac{\pi}{3}\right) \)  
   d \( f(x) = 2 \tan \left(2x + \frac{\pi}{3}\right) - 2 \)
1.2 Solving right-angled triangles

Pythagoras’ theorem

This well-known theorem is applicable to right-angled triangles and will be stated here without proof:

\[(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2\]

**Example 10**

In triangle \(ABC\), \(\angle ABC = 90^\circ\) and \(\angle CAB = x^\circ\),
\(AB = 6\) cm and \(BC = 5\) cm.

Find:

a. \(AC\)

b. the trigonometric ratios related to \(x^\circ\)

c. \(x\)

**Solution**

\[a. \quad \text{By Pythagoras’ theorem, } AC^2 = 5^2 + 6^2 = 61\]
\[\therefore \quad AC = \sqrt{61}\text{ cm}\]

\[b. \quad \sin x^\circ = \frac{5}{\sqrt{61}} \quad \cos x^\circ = \frac{6}{\sqrt{61}} \quad \tan x^\circ = \frac{5}{6}\]

\[c. \quad \tan x^\circ = \frac{5}{6}\]
\[\therefore \quad x = 39.81\text{ (correct to two decimal places)}\]

**Exercise 1B**

1. Find the trigonometric ratios \(\tan x^\circ\), \(\cos x^\circ\) and \(\sin x^\circ\) for each of the following triangles:

   a.  
      \[
      \begin{align*}
      &\text{a} \\
      &\text{x}^\circ \\
      &5 \\
      &8 \\
      \end{align*}
      \]

   b.  
      \[
      \begin{align*}
      &\text{b} \\
      &\text{x}^\circ \\
      &5 \\
      &7 \\
      \end{align*}
      \]

   c.  
      \[
      \begin{align*}
      &\text{c} \\
      &9 \\
      &7 \\
      \end{align*}
      \]

2. Find the exact value of \(a\) in each of the following triangles:

   a.  
      \[
      \begin{align*}
      &\text{a} \\
      &12 \\
      &30^\circ \\
      \end{align*}
      \]

   b.  
      \[
      \begin{align*}
      &\text{b} \\
      &45^\circ \\
      &6 \\
      \end{align*}
      \]

   c.  
      \[
      \begin{align*}
      &\text{c} \\
      &5 \\
      \end{align*}
      \]
3 Find the exact value of the pronumerals for each of the following:

4 a Find the values of \(a, y, z, w\) and \(x\).
   b Hence deduce exact values for \(\sin(15^\circ), \cos(15^\circ)\)
      and \(\tan(15^\circ)\).
   c Find the exact values for \(\sin(75^\circ), \cos(75^\circ)\) and
      \(\tan(75^\circ)\).

1.3 The sine and cosine rules

The sine rule

In section 1.2, methods for finding unknown lengths and angles for right-angled triangles were discussed. This section discusses methods for finding unknown quantities in non-right-angled triangles.

The sine rule is used to find unknown quantities in a triangle when one of the following situations arises:

- one side and two angles are given
- two sides and a non-included angle are given

In the first case a unique triangle is defined, but for the second it is possible for two triangles to exist.

Labelling convention

The following convention is followed in the remainder of this chapter. Interior angles are denoted by upper-case letters, and the length of the side opposite an angle is denoted by the corresponding lower-case letter.

The magnitude of angle \(BAC\) is denoted by \(A\).
The length of side \(BC\) is denoted by \(a\).
The sine rule states that for triangle $ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

A proof will only be given for the acute-angled triangle case. The proof for obtuse-angled triangles is similar.

**Proof**

In triangle $ACD$, \[ \sin A = \frac{h}{b} \] \[ \therefore h = b \sin A \]

In triangle $BCD$, \[ \sin B = \frac{h}{a} \] \[ \therefore h = a \sin B \] \[ \therefore a \sin B = b \sin A \] \[ \text{i.e.} \quad \frac{a}{\sin A} = \frac{b}{\sin B} \]

Similarly, starting with a perpendicular from $A$ to $BC$ would give

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

**Example 11**

Use the sine rule to find the length of $AB$.

**Solution**

$$\frac{c}{\sin 31^\circ} = \frac{10}{\sin 70^\circ}$$

\[ \therefore c = \frac{10 \times \sin 31^\circ}{\sin 70^\circ} \]

\[ \therefore c = 5.4809 \ldots \]

The length of $AB$ is 5.48 cm correct to two decimal places.

**Example 12**

Use the sine rule to find the magnitude of angle $XZY$, given that $Y = 25^\circ$, $y = 5$ and $z = 6$.
Solution
\[ \frac{5}{\sin 25^\circ} = \frac{6}{\sin Z} \]
\[ \therefore \quad \frac{6}{\sin Z} = \frac{5}{\sin 25^\circ} = 0.5071 \ldots \]
\[ \therefore \quad Z = \sin^{-1}(0.5071 \ldots) \]
\[ \therefore \quad Z = 30.4736 \ldots \text{ or } 180 - 30.4736 \ldots \]
\[ \therefore \quad Z = 30.47^\circ \text{ or } Z = 149.53^\circ \text{ correct to two decimal places.} \]

Remember: \( \sin(180 - \theta) = \sin \theta \)

There are two solutions for the equation \( \sin Z = 0.5071 \ldots \)

Note: When using the sine rule in the situation where two sides and a non-included angle are given, the possibility of two such triangles existing must be considered. Existence can be checked through the sum of the given angle and the calculated angle not exceeding 180°.

The cosine rule

The cosine rule is used to find unknown quantities in a triangle when one of the following situations arises:
- two sides and an included angle are given
- three sides are given

The cosine rule states that for triangle \( ABC \)
\[ a^2 = b^2 + c^2 - 2bc \cos A \]
equivalently, \( \cos A = \frac{b^2 + c^2 - a^2}{2bc} \)

The symmetrical results also hold, i.e.
- \( b^2 = a^2 + c^2 - 2ac \cos B \)
- \( c^2 = a^2 + b^2 - 2ab \cos C \)

The result will be proved for acute-angled triangles. The proof for obtuse-angled triangles is similar.

Proof

In triangle \( ACD \)
\[ b^2 = x^2 + h^2 \text{ (Pythagoras' theorem)} \]
\[ \cos A = \frac{x}{b} \text{ and therefore } x = b \cos A \]

In triangle \( BCD \)
\[ a^2 = (c - x)^2 + h^2 \text{ (Pythagoras' theorem)} \]
Expanding gives

\[ a^2 = c^2 - 2cx + x^2 + h^2 \]

\[ = c^2 - 2cx + b^2 \] (as \( x^2 + h^2 = b^2 \))

\[ \therefore a^2 = b^2 + c^2 - 2bc \cos A \] (as \( x = b \cos A \))

---

**Example 13**

For triangle \( ABC \), find the length of \( AB \) in centimetres correct to two decimal places.

**Solution**

\[ c^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos 67^\circ \]

\[ = 85.9268 \ldots \]

\[ \therefore c \approx 9.27 \]

The length of \( AB \) is 9.27 cm correct to two decimal places.

---

**Example 14**

Find the magnitude of angle \( ABC \) for triangle \( ABC \) correct to two decimal places.

**Solution**

\[ \cos B = \frac{a^2 + c^2 - b^2}{2ac} \]

\[ = \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6} \]

\[ = -0.3125 \]

\[ \therefore B = (108.2099 \ldots)^\circ \]

i.e. \( B \approx 108.21^\circ \) correct to two decimal places.

The magnitude of angle \( ABC \) is \( 108^\circ \ 12' \ 36'' \) (to the nearest second).

---

**Example 15**

In \( \triangle ABC \), \( \angle CAB = 82^\circ \), \( AC = 12 \) cm, \( AB = 15 \) cm. Find, correct to two decimal places:

a. \( BC \)  
b. \( \angle ACB \)
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Solution

a  BC is found by applying the cosine rule:
\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ = 12^2 + 15^2 - 2 \times 12 \times 15 \cos 82^\circ \]
\[ = 144 + 225 - 360 \times \cos 82^\circ \]
\[ = 318.897 \ldots \]

BC = a = 17.86 cm, correct to two decimal places.

b  \( \angle ACB \) is found by applying the sine rule pair:
\[ \frac{a}{\sin A} = \frac{c}{\sin C} \]
\[ \therefore \sin C = \frac{c \sin A}{a} \]
\[ = \frac{15 \times \sin 82^\circ}{17.86} \]
\[ \therefore \angle ACB = 56.28^\circ, \text{ correct to two decimal places.} \]

Note: 123.72° is also a solution to this equation but it is discarded as a possible answer as it is inconsistent with the information already given.

Exercise 1C

1  In triangle \( \triangle ABC \), \( \angle BAC = 73^\circ \), \( \angle ACB = 55^\circ \) and \( AB = 10 \) cm. Find, correct to two decimal places:
   a  BC  b  AC

2  In triangle \( \triangle ABC \), \( \angle ABC = 58^\circ \), \( AB = 6.5 \) cm and \( BC = 8 \) cm. Find, correct to two decimal places:
   a  AC  b  \( \angle BCA \)

3  The adjacent sides of a parallelogram are 9 cm and 11 cm. One of its angles is 67°. Find the length of the longer diagonal, correct to two decimal places.

4  In \( \triangle ABC \), \( \angle ACB = 34^\circ \), \( AC = 8.5 \) cm and \( AB = 5.6 \) cm. Find, correct to two decimal places:
   a  the two possible values of \( \angle ABC \) (one acute and one obtuse)
   b  BC in each case

5  In \( \triangle ABC \), \( \angle ABC = 35^\circ \), \( AB = 10 \) cm and \( BC = 4.7 \) cm. Find, correct to two decimal places:
   a  AC  b  \( \angle ACB \)

6  In \( \triangle ABC \), \( \angle ABC = 45^\circ \), \( \angle ACB = 60^\circ \) and \( AC = 12 \) cm. Find AB.
7. In $\triangle PQR$, $\angle QPR = 60^\circ$, $PQ = 2\, \text{cm}$ and $PR = 3\, \text{cm}$. Find $QR$.

8. In $\triangle ABC$, $\angle ABC$ has magnitude $40^\circ$, $AC = 20\, \text{cm}$ and $AB = 18\, \text{cm}$. Find the distance $BC$ correct to 2 decimal places.

9. In $\triangle ABC$, $\angle ACB$ has magnitude $30^\circ$, $AC = 10\, \text{cm}$ and $AB = 8\, \text{cm}$. Find the distance $BC$ using the cosine rule.

10. In $\triangle ABC$, $AB = 5\, \text{cm}$, $BC = 12\, \text{cm}$ and $AC = 10\, \text{cm}$. Find:
    a. the magnitude of $\angle ABC$, correct to two decimal places
    b. the magnitude of $\angle BAC$, correct to two decimal places

1.4 Geometry prerequisites

In the Specialist Mathematics study design it is stated that students should be familiar with several geometric results and be able to apply them in examples.

These results have been proved in earlier years’ study. In this section they are listed.

- The sum of the interior angles of a triangle is $180^\circ$.

$$a + b + c = 180$$

- The sum of the exterior angles of a convex polygon is $360^\circ$.

$$a + b + c + d + e = 360 \quad x + y + z + w = 360 \quad a + b + c + d + e + f = 360$$

- Corresponding angles of lines cut by a transversal are equal if, and only if, the lines are parallel.

  $a = b$ and $c = d$

  $a$ and $b$ are corresponding angles

  $c$ and $d$ are corresponding angles
Opposite angles of a parallelogram are equal, and opposite sides are equal in length.

\[ AB = DC \quad AD = BC \]

\[ c = d \quad \text{and} \quad a = b \]

The base angles of an isosceles triangle are equal.

\((AB = AC)\)

\[ a = b \]

The line joining the vertex to the midpoint of the base of an isosceles triangle is perpendicular to the base.

The perpendicular bisector of the base of an isosceles triangle passes through the opposite vertex.

The angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference.

\[ x = 2y \]

The angle in a semicircle is a right angle.

Angles in the same segment of a circle are equal.

\[ x = y = z \]

The sum of the opposite angles of a cyclic quadrilateral is 180°.

\[ x + y = 180 \quad z + w = 180 \]

\[ a = x \]

An exterior angle of a cyclic quadrilateral and the interior opposite angle are equal.
A tangent to a circle is perpendicular to the radius at the point of contact.

The two tangents to a circle from an exterior point are equal in length.

An angle between a tangent to a circle and a chord through the point of contact is equal to the angle in the alternate segment.

**Example 16**

Find the magnitude of each of the following angles:

- a \( \angle ABC \)
- b \( \angle ADC \)
- c \( \angle CBD \)
- d \( \angle OCD \)
- e \( \angle BAD \)

**Solution**

- a \( \angle ABC = 93^\circ \) (vertically opposite)
- b \( \angle ADC = 87^\circ \) (opposite angle of a cyclic quadrilateral)
- c \( \angle COB = 85^\circ \) (vertically opposite)
- \( \angle CBD = [180 - (60 + 85)]^\circ = 35^\circ \) (angles of a triangle, \( \triangle CBO \))
- d \( \angle CAD = 35^\circ \) (angle subtended by the arc \( CD \))
- \( \angle ADC = 87^\circ \) (From b)
- \( \angle OCD = [180 - (87 + 35)]^\circ = 58^\circ \) (angles of a triangle, \( \triangle CAD \))
- e \( \angle BAD = [180 - (60 + 58)]^\circ = 62^\circ \) (opposite angles of a cyclic quadrilateral)

**Exercise 1D**

1 Find the value of \( a, y, z \) and \( x \).
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2 Find the magnitude of each of the following:
   a $\angle RTW$
   b $\angle TSW$
   c $\angle TRS$
   d $\angle RWT$

3 Find the value of $a$, $b$ and $c$. $AB$ is a tangent to the circle at $C$.

4 $ABCD$ is a square and $ABX$ is an equilateral triangle.
   Find the magnitude of:
   a $\angle DXC$
   b $\angle XDC$

5 Find the values of $a$, $b$, $c$, $d$ and $e$.

6 Find $x$ in terms of $a$, $b$ and $c$.

7 Find the values of $x$ and $y$, given that $O$ is the center of the circle.

8 Find the values of $a$, $b$, $c$ and $d$.

9 Find the values of $x$ and $y$.

10 $O$ is the centre of the circle. Find the values of $x$ and $y$. 
1.5 Sequences and series

The following are examples of sequences of numbers:

- $a$ 1, 3, 5, 7, 9, ...
- $b$ 0.1, 0.11, 0.111, 0.1111, ...
- $c$ $rac{1}{3}$, $rac{1}{9}$, $rac{1}{27}$, $rac{1}{81}$, ...
- $d$ 10, 7, 4, 1, −2, ...
- $e$ 0.6, 1.7, 2.8, 3.9, ...

Note that each sequence is a set of numbers, with order being important.

For some sequences of numbers a rule can be found connecting any number to the preceding number. For example:

- For sequence A, a rule is: add 2
- For sequence C, a rule is: multiply by $\frac{1}{3}$
- For sequence D, a rule is: subtract 3
- For sequence E, a rule is: add 1.1

The numbers of a sequence are called terms. The $n$th term of a sequence is denoted by the symbol $t_n$. So the first term is $t_1$, the 12th term is $t_{12}$ and so on.

A sequence can be defined by specifying a rule which enables each subsequent term to be found using the previous term. In this case, the rule specified is called an iterative rule or a difference equation. For example:

- sequence A can be defined by $t_1 = 1$, $t_n = t_{n-1} + 2$
- sequence C can be defined by $t_1 = \frac{1}{3}$, $t_n = \frac{1}{3}t_{n-1}$

Example 17

Use the difference equation to find the first four terms of the sequence $t_1 = 3$, $t_n = t_{n-1} + 5$

**Solution**

$t_1 = 3$

$t_2 = t_1 + 5 = 8$

$t_3 = t_2 + 5 = 13$

$t_4 = t_3 + 5 = 18$

The first four terms are 3, 8, 13, 18.

Example 18

Find the difference equation for the following sequence.

$9, -3, 1, -\frac{1}{3}, ...$

**Solution**

$-3 = -\frac{1}{3} \times 9$ i.e. $t_2 = -\frac{1}{3}t_1$

$1 = -\frac{1}{3} \times -3$ i.e. $t_3 = -\frac{1}{3}t_2$

$\therefore t_n = -\frac{1}{3}t_{n-1}, t_1 = 9$
Alternatively, a sequence can be defined by a rule that is stated in terms of $n$. For example:

$t_n = 2n$ defines the sequence $t_1 = 2, t_2 = 4, t_3 = 6, t_4 = 8 \ldots$

$t_n = 2^{n-1}$ defines the sequence $t_1 = 1, t_2 = 2, t_3 = 4, t_4 = 8 \ldots$

**Example 19**

Find the first four terms of the sequence defined by the rule $t_n = 2n + 3$.

**Solution**

\[
\begin{align*}
  t_1 &= 2 \times 1 + 3 = 5 \\
  t_2 &= 2 \times 2 + 3 = 7 \\
  t_3 &= 2 \times 3 + 3 = 9 \\
  t_4 &= 2 \times 4 + 3 = 11 \\
\end{align*}
\]

The first four terms are 5, 7, 9, 11.

**Arithmetic sequences**

A sequence in which each successive term is found by adding a constant value to the previous term is called an **arithmetic sequence**. For example, 2, 5, 8, 11 \ldots is an arithmetic sequence.

An arithmetic sequence can be defined by a difference equation of the form

\[ t_n = t_{n-1} + d \] where $d$ is a constant.

If the first term of an arithmetic sequence $t_1 = a$ then the $n$th term of the sequence can also be described by the rule:

\[ t_n = a + (n - 1)d \] where $a = t_1$ and $d = t_n - t_{n-1}$

$d$ is the **common difference**.

**Example 20**

Find the tenth term of the arithmetic sequence $-4, -1, 2, 5 \ldots$

**Solution**

\[
\begin{align*}
  a &= -4, d = 3, n = 10 \\
  t_n &= a + (n - 1)d \\
  t_{10} &= -4 + (10 - 1)3 \\
  &= 23 \\
\end{align*}
\]

**Arithmetic series**

The sum of the terms in a sequence is called a **series**. If the sequence in question is arithmetic, the series is called an **arithmetic series**. The symbol $S_n$ is used to denote the sum of $n$ terms of a sequence,

\[ S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) \]

If this sum is written in reverse order, then

\[ S_n = (a+(n-1)d) + (a+(n-2)d) + \cdots + (a+d) + a \]
Adding these two expressions together gives $2S_n = n[2a + (n - 1)d]$

\[
\therefore S_n = \frac{n}{2}[2a + (n - 1)d]
\]

and since the last term \( l = t_n = a + (n - 1)d \)

\[
S_n = \frac{n}{2}(a + l)
\]

**Geometric sequences**

A sequence in which each successive term is found by multiplying the previous term by a fixed value is called a geometric sequence.

For example, 2, 6, 18, 54 . . . is a geometric sequence.

A geometric sequence can be defined by an iterative equation of the form \( t_n = rt_{n-1} \), where \( r \) is constant.

If the first term of a geometric sequence \( t_1 = a \), then the \( n \)th term of the sequence can also be described by the rule

\[
t_n = ar^{n-1} \quad \text{where} \quad r = \frac{t_n}{t_{n-1}}
\]

\( r \) is called the common ratio.

---

**Example 21**

Calculate the tenth term of the sequence 2, 6, 18 . . .

**Solution**

\[ a = 2, \; r = 3, \; n = 10 \]

\[ t_n = ar^{n-1} \]

\[ t_{10} = 2 \times 3^{(10-1)} = 39366 \]

**Geometric series**

The sum of the terms in a geometric sequence is called a geometric series. An expression for \( S_n \), the sum of \( n \) terms, of a geometric sequence can be found using a similar method to that used in the development of a formula for an arithmetic series.

Let \[ S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad \ldots [1] \]

Then \[ rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad \ldots [2] \]


\[ rS_n - S_n = ar^n - a \]

\[ \therefore S_n(r - 1) = a(r^n - 1) \]

and

\[ S_n = \frac{a(r^n - 1)}{r - 1} \]
For values of \( r \) such that \(-1 < r < 1\), it is often more convenient to use the alternative formula

\[
S_n = \frac{a(1 - r^n)}{1 - r}
\]

which is obtained by subtracting \( 2 \) from \( 1 \) above.

### Example 22

Find the sum of the first nine terms of the geometric sequence \( \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81} \ldots \)

#### Solution

\[
a = \frac{1}{3}, \quad r = \frac{1}{3}, \quad n = 9
\]

\[
S_9 = \frac{\frac{1}{3}(\left(\frac{1}{3}\right)^9 - 1)}{\frac{1}{3} - 1}
\]

\[
= \frac{1}{2}\left(\left(\frac{1}{3}\right)^9 - 1\right)
\]

\[
\approx \frac{1}{2}(0.999949)
\]

\[
\approx 0.499975 \text{ (to 6 decimal places)}
\]

### Infinite geometric series

If the common ratio of a geometric sequence has a magnitude less than 1, i.e. \(-1 < r < 1\), then each successive term of the sequence is closer to zero e.g., \(4, 2, 1, \frac{1}{2}, \frac{1}{4} \ldots\)

When the terms of the sequence are added, the corresponding series \(a + ar + ar^2 + \cdots + ar^{n-1}\) will approach a limiting value, i.e. as \(n \to \infty\), \(S_n \to\) a limiting value. Such a series is called **convergent**.

In example 22 above, it was found that for the sequence \(\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81} \ldots\) the sum of the first nine terms, \(S_n\), was 0.499975.

For the same sequence, \(S_{20} = 0.499999999 \approx 0.5\)

So even for a relatively small value of \(n\) (20), the sum approaches the limiting value of 0.5 very quickly.

Given that \(S_n = \frac{a(1 - r^n)}{1 - r}\)

\[
\Rightarrow S_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r}
\]

as \(n \to \infty\), \(r^n \to 0\) and hence \(\frac{ar^n}{1 - r} \to 0\)

It follows then that the limit as \(n \to \infty\) of \(S_n\) is \(\frac{a}{1 - r}\)

So

\[
S_\infty = \frac{a}{1 - r}
\]

This is also referred to as ‘the sum to infinity’ of the series.
Example 23

Find the sum to infinity of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

Solution

$r = \frac{1}{2}, \ a = 1$

$\therefore \ S_\infty = \frac{1}{1 - \frac{1}{2}} = 2$

Using a graphics calculator

Example 24

Graph the terms of the geometric sequence defined by:

$a_n = 512(0.5)^{n-1}$ for $n = 1, 2\ldots$

Solution

Step 1  Set your calculator to sequence mode.

a  Enter the MODE menu; move cursor down to line 4 and across to cover Seq.

b  Press ENTER to select sequence mode.

Step 2  Enter expression defining sequence.

Press Y= and enter:

a  set nMin = 1

b  $u(n) = 512(0.5) \ ^{(n - 1)}$  

(\textbf{Note:} $n$ is obtained by pressing XT\textup{\textregistered}N)

c  $u(nMin) = 512$

Step 3  Set viewing window.

Press WINDOW and enter the values as shown.

Step 4  Plot.

Press GRAPH.

Note that terms of the sequence can be viewed in the TABLE window.
Example 25

a Generate the first 10 terms in the arithmetic sequence whose \( n \)th term is 
\[ 8 + 4(n - 1) \] 
and store them in list \( L1 \).
b Find the sum of these first 10 terms.
c This arithmetic sequence can also be generated by the recursive form, 
\[ t_n = t_{n-1} + 4, \] 
with \( t_1 = 8 \). Generate the first 10 terms in this way

Solution

a Select Seq mode from the MODE menu.
In the \( Y= \) window enter
\[ nMin = 1, \]
\[ u(n) = 8 + 4(n - 1), \]
and
\[ u(nMin) = 8 \]
Press \( STAT \) and select 1:Edit from the EDIT menu.
Take the cursor to the top of \( L2 \) (actually over \( L2 \)) and 
press \( \text{ENTER} \). In the entry line enter
\[ L2 = "8 + 4(L1 - 1)" \] 
The result is as shown.
b Take the cursor to \( L3 \) and enter
\[ L3 = "\text{cumSum}(L2)" \] 
and press \( \text{ENTER} \). The result is as shown. \text{cumSum} is obtained by selecting 6:
\text{cumSum} from the OPS submenu of LIST.
c In the \( Y= \) window enter, \( nMin = 1, \)
\[ u(n) = u(n - 1) + 4 \] 
and \( u(nMin) = 8 \). Note that \( u \) is 
obtained by pressing \( 2 \text{ND} \) \( 7 \) and \( n \) is obtained by 
pressing \( \text{X}\theta N \). The sequence can be seen in the TABLE screen.

Using a CAS calculator

The method is very similar to that discussed above.
Press \( \text{MODE} \) and use the right arrow from Graph to 
reveal the Graph menu. Select 4:SEQUENCE and 
press \( \text{ENTER} \).
In the Y= screen enter
\[ u_1(n) = u_1(n - 1) + 4 \]
and \( u_1 = 8 \). The sequence can be seen in the TABLE window.

**Exercise 1E**

1. A difference equation has rule \( t_{n+1} = 3t_n - 1 \), \( t_1 = 6 \). Find \( t_2 \) and \( t_3 \). Use a graphics calculator to find \( t_8 \).

2. A difference equation has rule \( y_{n+1} = 2y_n + 6 \), \( y_1 = 5 \). Find \( y_2 \) and \( y_3 \). Use a graphics calculator to find \( y_{10} \) and to plot a graph showing the first ten values.

3. The Fibonacci sequence is given by the difference equation \( t_{n+2} = t_{n+1} + t_n \) where \( t_1 = t_2 = 1 \). Find the first ten terms of the Fibonacci sequence.

4. Find the sum of the first ten terms of an arithmetic sequence with first term 3 and common difference 4.

5. Find the sum to infinity of \( 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots \)

6. The first, second and third terms of a geometric sequence are \( x + 5 \), \( x \) and \( x - 4 \) respectively. Find:
   - (a) \( x \)
   - (b) the common ratio
   - (c) the difference between the sum to infinity and the sum of the first ten terms

7. Find the sum of the first eight terms of a geometric sequence with first term 6 and common ratio \(-3\).

8. Find the sum to infinity of the geometric sequence \( a, \frac{a}{\sqrt{2}}, \frac{a}{2}, \frac{a}{2\sqrt{2}} \ldots \) in terms of \( a \).

9. Consider the sum \( S_n = 1 + \frac{x}{2} + \frac{x^2}{4} + \cdots + \frac{x^{n-1}}{2^{n-1}} \)
   - (a) Calculate \( S_{10} \) when \( x = 1.5 \).
   - (b) i Find the possible values of \( x \) for which the infinite sum exists. Denote this sum by \( S \).
   - ii Find the values of \( x \) for which \( S = 2S_{10} \).

10. a Find an expression for the infinite geometric sum \( 1 + \sin \theta + \sin^2 \theta + \ldots \)
    b Find the values of \( \theta \) for which the infinite geometric sum is 2.

### 1.6 Circles

For a circle with centre the origin and radius \( r \), if the point with coordinates \((x, y)\) is on the circle then \( x^2 + y^2 = r^2 \).
The converse is also true, i.e. a point with coordinates \((x, y)\) such that \(x^2 + y^2 = r^2\) lies on the circle with centre the origin and radius \(r\).

Applying Pythagoras’ theorem to triangle \(OAP\) yields

\[
r^2 = OP^2 = x^2 + y^2
\]

In general, the following result holds:

The circle with centre \((h, k)\) and radius \(r\) is the graph of the equation

\[
(x - h)^2 + (y - k)^2 = r^2
\]

This graph is obtained from the graph of \(x^2 + y^2 = r^2\) by the translation defined by

\[
(x, y) \rightarrow (x + h, y + k)
\]

**Example 26**

Sketch the graph of the circle with centre at \((-2, 5)\) and radius 2, and state the cartesian equation for this circle.

**Solution**

The equation is

\[
(x + 2)^2 + (y - 5)^2 = 4
\]

which may also be written as

\[
x^2 + y^2 + 4x - 10y + 25 = 0
\]

**Note:** The equation \(x^2 + y^2 + 4x - 10y + 25 = 0\) can be ‘unsimplified’ by completing the square.

\[
x^2 + y^2 + 4x - 10y + 25 = 0
\]

implies \(x^2 + 4x + 4 + y^2 - 10y + 25 + 25 = 29\)

i.e. \((x + 2)^2 + (y - 5)^2 = 4\)

This suggests a general form of the equation of a circle.

\[
x^2 + y^2 + Dx + Ey + F = 0
\]

Completing the square gives

\[
x^2 + Dx + \left(\frac{D^2}{4}\right) + y^2 + Ey + \left(\frac{E^2}{4}\right) + F = \frac{D^2 + E^2}{4}
\]

i.e.

\[
\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 + E^2 - 4F}{4}
\]
If \( D^2 + E^2 - 4F > 0 \), then the equation represents a circle with centre \( \left( \frac{-D}{2}, \frac{-E}{2} \right) \) and radius \( \sqrt{\frac{D^2 + E^2 - 4F}{4}} \).

If \( D^2 + E^2 - 4F = 0 \), then the equation represents one point \( \left( \frac{-D}{2}, \frac{-E}{2} \right) \).

If \( D^2 + E^2 - 4F < 0 \), then the equation has no graphical representation in the cartesian plane.

**Example 27**

Sketch the graph of \( x^2 + y^2 + 4x + 6y - 12 = 0 \). State the coordinates of the centre and the radius.

**Solution**

\[
\begin{align*}
\quad x^2 + y^2 + 4x + 6y - 12 &= 0 \\
\therefore x^2 + 4x + 4 + y^2 + 6y + 9 - 12 &= 13 \\
i.e. \quad (x + 2)^2 + (y + 3)^2 &= 25
\end{align*}
\]

The circle has centre \((-2, -3)\) and radius 5.

**Example 28**

Sketch a graph of the region of the plane such that \( x^2 + y^2 < 9 \) and \( x \geq 1 \).

**Solution**

\[
\begin{align*}
\text{required region}
\end{align*}
\]

**Exercise 1F**

1. Find the equations of the circles with the following centres and radii:
   - a centre (2, 3); radius 1
   - b centre (−3, 4); radius 5
   - c centre (0, −5); radius 5
   - d centre (3, 0); radius \( \sqrt{2} \)

2. Find the radii and the coordinates of the centres of the circles with the following equations:
   - a \( x^2 + y^2 + 4x - 6y + 12 = 0 \)
   - b \( x^2 + y^2 - 2x - 4y + 1 = 0 \)
   - c \( x^2 + y^2 - 3x = 0 \)
   - d \( x^2 + y^2 + 4x - 10y + 25 = 0 \)
3 Sketch the graphs of each of the following:
   a \(2x^2 + 2y^2 + x + y = 0\)  
   b \(x^2 + y^2 + 3x - 4y = 6\)  
   c \(x^2 + y^2 + 8x - 10y + 16 = 0\)  
   d \(x^2 + y^2 - 8x - 10y + 16 = 0\)  
   e \(2x^2 + 2y^2 - 8x + 5y + 10 = 0\)  
   f \(3x^2 + 3y^2 + 6x - 9y = 100\)

4 Sketch the graphs of the regions of the plane specified by the following:
   a \(x^2 + y^2 \leq 16\)  
   b \(x^2 + y^2 \geq 9\)  
   c \((x - 2)^2 + (y - 2)^2 < 4\)  
   d \((x - 3)^2 + (y + 2)^2 > 16\)  
   e \(x^2 + y^2 \leq 16\) and \(x \leq 2\)  
   f \(x^2 + y^2 \leq 9\) and \(y \geq -1\)

5 The points \((8, 4)\) and \((2, 2)\) are the ends of a diameter of a circle. Find the coordinates of the centre and the radius of the circle.

6 Find the equation of the circle, centre \((2, -3)\), which touches the \(x\) axis.

7 Find the equation of the circle which passes through \((3, 1)\), \((8, 2)\) and \((2, 6)\).

8 Find the radii and coordinates of the centre of the circles with equations
   \(4x^2 + 4y^2 - 60x - 76y + 536 = 0\) and \(x^2 + y^2 - 10x - 14y + 49 = 0\), and find the coordinates of the points of intersection of the two curves.

9 Find the coordinates of the points of intersection of the circle with equation \(x^2 + y^2 = 25\) and the line with equation:
   a \(y = x\)  
   b \(y = 2x\)

1.7 Ellipses and hyperbolas

Ellipses

The curve with equation \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) is an ellipse with centre the origin, \(x\)-axis intercepts \((-a, 0)\) and \((a, 0)\) and \(y\)-axis intercepts \((0, -b)\) and \((0, b)\).

If \(a > b\) the ellipse will appear as shown in the diagram on the left.

If \(b > a\) the ellipse is as shown in the diagram on the right.

If \(b = a\), the equation is that of a circle with centre the origin and radius \(a\).
The general cartesian form is as given below.

The curve with equation

\[
\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1
\]

is an ellipse with centre \((h, k)\). It is obtained by a translation of the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\).

The translation is \((x, y) \rightarrow (x + h, y + k)\).

The ellipse with equation \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) can be obtained by applying the following dilations to the circle with equation \(x^2 + y^2 = 1\):

- a dilation of factor \(a\) from the \(y\) axis, i.e. \((x, y) \rightarrow (ax, y)\)
- a dilation of factor \(b\) from the \(x\) axis, i.e. \((x, y) \rightarrow (x, by)\)

The result is the transformation \((x, y) \rightarrow (ax, by)\).

Example 29

Sketch the graph of each of the following. Give the axes intercepts and the coordinates of the centre.

a \(\frac{x^2}{9} + \frac{y^2}{4} = 1\)  

b \(\frac{x^2}{4} + \frac{y^2}{9} = 1\)

c \(\frac{(x - 2)^2}{9} + \frac{(y - 3)^2}{16} = 1\)  

d \(3x^2 + 24x + y^2 + 36 = 0\)
Solution

a

Centre (0, 0)
Axes intercepts (±3, 0) and (0, ±2)

b

Centre (0, 0)
Axes intercepts (±2, 0) and (0, ±3)

Centre is at (2, 3)

When \( x = 0 \)

\[
\frac{4}{9} + \frac{(y - 3)^2}{16} = 1
\]

\[
\therefore \frac{(y - 3)^2}{16} = \frac{5}{9}
\]

\[
\therefore (y - 3)^2 = \frac{16 \times 5}{9}
\]

\[
\therefore y = 3 \pm \frac{4\sqrt{5}}{3}
\]

When \( y = 0 \)

\[
\frac{(x - 2)^2}{9} + \frac{9}{16} = 1
\]

\[
\therefore \frac{(x - 2)^2}{9} = \frac{7}{16}
\]

\[
\therefore (x - 2)^2 = \frac{7 \times 16}{9}
\]

\[
\therefore x = 2 \pm \frac{3\sqrt{7}}{4}
\]

c

d

\[3x^2 + 24x + y^2 + 36 = 0\]
Completing the square yields

\[3[x^2 + 8x + 16] + y^2 + 36 - 48 = 0\]

i.e.

\[3(x + 4)^2 + y^2 = 12\]

\[
\frac{(x + 4)^2}{4} + \frac{y^2}{12} = 1
\]

\[
\therefore Centre (-4, 0)
\]
Axes intercepts (−6, 0) and (−2, 0)
Defining an ellipse

In the previous section a circle was defined as a set of points which are all a constant distance from a given point (the centre). An ellipse can be defined in a similar way.

Consider the set of all points \( P \) such that \( PF_1 + PF_2 \) is equal to a constant \( k \) with \( k > 2m \), and the coordinates of \( F_1 \) and \( F_2 \) are \((m, 0)\) and \((-m, 0)\) respectively. We can show that the equation describing this set of points is

\[
\frac{x^2}{a^2} + \frac{y^2}{a^2 - m^2} = 1 \quad \text{where} \quad k = 2a.
\]

This can be pictured as a string of length \( P_1F_1 + P_1F_2 \) being attached by nails to a board at \( F_1 \) and \( F_2 \) and, considering the path mapped out by a pencil, extending the string so that it is taut, and moving ‘around’ the two points.

Let the coordinates of \( P \) be \((x, y)\).

\[
PF_1 = \sqrt{(x - m)^2 + y^2} \quad \text{and} \quad PF_2 = \sqrt{(x + m)^2 + y^2}
\]

and assume

\[
PF_1 + PF_2 = k
\]

Then

\[
\sqrt{(x + m)^2 + y^2} + \sqrt{(x - m)^2 + y^2} = k
\]

Rearranging and squaring gives

\[
(x + m)^2 + y^2 = k^2 - 2k\sqrt{(x - m)^2 + y^2} + (x + m)^2 + y^2
\]

\[
\therefore \quad 4mx = k^2 - 2k\sqrt{(x - m)^2 + y^2}
\]

Rearranging and squaring again gives

\[
4k^2(x - m)^2 + 4k^2y^2 = k^4 - 8k^2mx + 16m^2x^2
\]

Collecting like terms

\[
4(k^2 - 4m^2)x^2 + 4k^2y^2 = k^2(k^2 - 4m^2)
\]

\[
\therefore \quad \frac{4x^2}{k^2} + \frac{4y^2}{k^2 - 4m^2} = 1
\]

Let \( a = \frac{k}{2} \), then \( \frac{x^2}{a^2} + \frac{y^2}{a^2 - m^2} = 1 \)
The points $F_1$ and $F_2$ are called the **foci** of the ellipse. The constant $k = 2a$ is called the **focal sum**.

If $a = 3$ and $m = 2$, the ellipse with equation $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is obtained.

For an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $a > b$, the foci are at $(\pm \sqrt{a^2 - b^2}, 0)$.

Given an equation of the form $Ax^2 + By^2 + Cx + Ey + F = 0$, where $A$ and $B$ are both positive (or both negative), the corresponding graph is an ellipse or a point. If $A = B$ the graph is that of a circle. In some cases, as for the circle, no pairs $(x, y)$ will satisfy the equation.

### Hyperbolas

The curve with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola with centre at the origin. The axis intercepts are $(a, 0)$ and $(-a, 0)$.

The hyperbola has asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$. An informal argument for this is as follows.

The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be rearranged:

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$\therefore \quad y^2 = \frac{b^2}{a^2} \left(1 - \frac{a^2}{x^2}\right)$$

But as $x \to \pm \infty$, $\frac{a^2}{x^2} \to 0$

$$\therefore \quad y^2 \to \frac{b^2}{a^2}$$

i.e. $y \to \pm \frac{bx}{a}$

The general equation for a hyperbola is formed by suitable translations.

The curve with equation

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

is a hyperbola with centre $(h, k)$. The asymptotes are

$$y - k = \pm \frac{b}{a}(x - h)$$

This hyperbola is obtained from the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by the translation defined by $(x, y) \to (x + h, y + k)$. 

Example 30

For each of the following equations, sketch the graph of the corresponding hyperbola, give the coordinates of the centre and the axes intercepts, and the equations of the asymptotes.

a \[ \frac{x^2}{9} - \frac{y^2}{4} = 1 \]

b \[ \frac{y^2}{9} - \frac{x^2}{4} = 1 \]

c \[ (x - 1)^2 - (y + 2)^2 = 1 \]

d \[ \frac{(y - 1)^2}{4} - \frac{(x + 2)^2}{9} = 1 \]

Solution

a \[ \frac{x^2}{9} - \frac{y^2}{4} = 1 \]

\[ \therefore \quad y^2 = \frac{4x^2}{9} \left( 1 - \frac{9}{x^2} \right) \]

Equation of asymptotes:

\[ y = \pm \frac{2}{3}x \]

When \( y = 0 \), \( x^2 = 9 \) and therefore \( x = \pm 3 \).

Axes intercepts (3, 0) and (−3, 0), centre (0, 0).

b \[ \frac{y^2}{9} - \frac{x^2}{4} = 1 \]

is the reflection of \[ \frac{x^2}{9} - \frac{y^2}{4} = 1 \] in the line \( y = x \).

\[ \therefore \text{asymptotes are} \quad x = \pm \frac{2}{3}y \]

i.e. \[ y = \pm \frac{3}{2} \]

The \( y \)-axis intercepts are (0, 3) and (0, −3).

c \[ (x - 1)^2 - (y + 2)^2 = 1 \]. The graph of \( x^2 - y^2 = 1 \) is sketched first. The asymptotes are \( y = x \) and \( y = -x \).

This hyperbola is called a **rectangular hyperbola** as its asymptotes are perpendicular. The centre is (0, 0) and the axes intercepts are at (1, 0) and (−1, 0).
A translation of \((x, y) \rightarrow (x + 1, y - 2)\) is applied. The new centre is \((1, -2)\) and the asymptotes have equations \(y + 2 = \pm (x - 1)\), i.e. \(y = x - 3\) and \(y = -x - 1\).

When \(x = 0\), \(y = -2\) and when \(y = 0\)

\[
(x - 1)^2 = 5
\]

\[
x = 1 \pm \sqrt{5}
\]

\[
\frac{(y - 1)^2}{4} - \frac{(x + 2)^2}{9} = 1
\]

is obtained by translating the hyperbola \(\frac{y^2}{4} - \frac{x^2}{9} = 1\) through the translation defined by \((x, y) \rightarrow (x - 2, y + 1)\).

\[
\frac{y^2}{4} - \frac{x^2}{9} = 1
\]

Note that the asymptotes for \(\frac{y^2}{4} - \frac{x^2}{9} = 1\) are the same as for those of the hyperbola \(\frac{x^2}{9} - \frac{y^2}{4} = 1\). The two hyperbolas are called **conjugate hyperbolas**.

**Defining a hyperbola**

Hyperbolas can be defined in a manner similar to the methods discussed earlier in this section for circles and ellipses.

Consider the set of all points, \(P\), such that \(PF_1 - PF_2 = k\) where \(k\) is a constant and \(F_1\) and \(F_2\) are points with coordinates \((m, 0)\) and \((-m, 0)\) respectively.

Then the equation of the curve defined in this way is

\[
\frac{x^2}{a^2} - \frac{y^2}{m^2 - a^2} = 1 \quad k = 2a
\]
Exercise 1G

1 Sketch the graph of each of the following. Label the axes intercepts. State the coordinates of the centre.

   a \( \frac{x^2}{9} + \frac{y^2}{16} = 1 \)
   b \( 25x^2 + 16y^2 = 400 \)
   c \( \frac{(x - 4)^2}{9} + \frac{(y - 1)^2}{16} = 1 \)
   d \( x^2 + \frac{(y - 2)^2}{9} = 1 \)
   e \( 9x^2 + 25y^2 - 54x - 100y = 44 \)
   f \( 9x^2 + 25y^2 = 225 \)
   g \( 5x^2 + 9y^2 + 20x - 18y - 16 = 0 \)
   h \( 16x^2 + 25y^2 - 32x + 100y - 284 = 0 \)
   i \( \frac{(x - 2)^2}{4} + \frac{(y - 3)^2}{9} = 1 \)
   j \( 2(x - 2)^2 + 4(y - 1)^2 = 16 \)

2 Sketch the graphs of each of the following. Label the axes intercepts and give the equations of the asymptotes.

   a \( \frac{x^2}{16} - \frac{y^2}{9} = 1 \)
   b \( \frac{y^2}{16} - \frac{x^2}{9} = 1 \)
   c \( x^2 - y^2 = 4 \)
   d \( 2x^2 - y^2 = 4 \)
   e \( x^2 - 4y^2 - 4x - 8y - 16 = 0 \)
   f \( 9x^2 - 25y^2 - 90x + 150y = 225 \)
   g \( \frac{(x - 2)^2}{4} - \frac{(y - 3)^2}{9} = 1 \)
   h \( 4x^2 - 8x - y^2 + 2y = 0 \)
   i \( 9x^2 - 16y^2 - 18x + 32y - 151 = 0 \)
   j \( 25x^2 - 16y^2 = 400 \)

3 Find the coordinates of the points of intersection of \( y = \frac{1}{2}x \) with:

   a \( x^2 - y^2 = 1 \)
   b \( \frac{x^2}{4} + y^2 = 1 \)

4 Show that there is no intersection point of the line \( y = x + 5 \) and the ellipse \( x^2 + \frac{y^2}{4} = 1 \).

5 Find the coordinates of the points of intersection of the curves \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) and \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \). Show that the points of intersection are the vertices of a square.

6 Find the coordinates of the points of intersection of \( \frac{x^2}{16} + \frac{y^2}{25} = 1 \) and the line with equation \( 5x = 4y \).

7 On the one set of axes sketch the graphs of \( x^2 + y^2 = 9 \) and \( x^2 - y^2 = 9 \).

8 Sketch each of the following regions:

   a \( x^2 - y^2 \leq 1 \)
   b \( x^2 - y^2 \geq 4 \)
   c \( y^2 \leq \frac{x^2}{4} - 1 \)
   d \( \frac{x^2}{9} + \frac{y^2}{4} < 1 \)
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\[ e \quad x^2 - y^2 \leq 1 \text{ and } x^2 + y^2 \leq 4 \]
\[ f \quad \frac{(x - 3)^2}{16} + \frac{y^2}{9} \leq 1 \]
\[ g \quad x^2 - y^2 \leq 4 \text{ and } \frac{x^2}{9} + y^2 \leq 1 \]
\[ h \quad x^2 - y^2 > 1 \text{ and } x^2 + y^2 \leq 4 \]
\[ i \quad \frac{(x - 2)^2}{9} + y^2 \leq 4 \]
\[ j \quad \frac{x^2}{4} + y^2 \leq 1 \text{ and } y \leq x \]

1.8 Parametric equations of circles, ellipses and hyperbolas

Circles

It is sometimes useful to express the rule of a relation in terms of a third variable, called a parameter. We have already seen in the work on circular functions that the unit circle can be expressed in cartesian form, i.e. \( \{(x, y): x^2 + y^2 = 1\} \) or in the form \( \{x, y): x = \cos t, y = \sin t, \text{ with } t \in [0, 2\pi]\} \). The latter is called the set of parametric equations of the unit circle which give the coordinates \((x, y)\) of all points on the unit circle.

The restriction for the values of \( t \) is unnecessary in the representation of the graph as \( \{x, y): x = \cos t, y = \sin t, \text{ with } t \in R\} \) gives the same points with repetitions since \( \cos(2\pi + t) = \cos t \text{ and } \sin(2\pi + t) = \sin t \). If the set of values for \( t \) is the interval \([0, \pi]\), only the top half of the circle is obtained.

The set notation is often omitted, and in the following this will be done. The next three diagrams illustrate the graphs resulting from the parametric equations \( x = \cos t \) and \( y = \sin t \) for three different sets of values of \( t \).

![Parametric equations of circles](image)

\[ t \in [0, 2\pi] \quad t \in [0, \pi] \quad t \in [0, \frac{\pi}{2}] \]

In general, \( x^2 + y^2 = a^2 \), where \( a > 0 \), is the cartesian equation of a circle with centre at the origin and radius \( a \). The parametric equations are \( x = a \cos t \) and \( y = a \sin t \). The minimal interval of \( t \) values to yield the entire circle is \([0, 2\pi]\).

The domain and range of the corresponding cartesian relation can be determined by the parametric equation determining the \( x \) value and the \( y \) value respectively. The range of the function with rule \( x = a \cos t, t \in [0, 2\pi] \) is \([-a, a]\) and hence the domain of the relation \( x^2 + y^2 = a^2 \) is \([-a, a]\). The range of the function with rule \( y = a \sin t, t \in [0, 2\pi] \) is \([-a, a]\) and hence the range of the relation \( x^2 + y^2 = a^2 \) is \([-a, a]\).
Example 31

A circle is defined by the parametric equations \( x = 2 + 3 \cos \theta \) and \( y = 1 + 3 \sin \theta \) for \( \theta \in [0, 2\pi] \). Find the corresponding cartesian equation of the circle and state the domain and range of this relation.

Solution

The range of the function with rule \( x = 2 + 3 \cos \theta \) is \([-1, 5]\) and hence the domain of the corresponding cartesian relation is \([-1, 5]\). The range of the function with rule \( y = 1 + 3 \sin \theta \) is \([-2, 4]\) and hence the range of the corresponding cartesian relation is \([-2, 4]\).

Rewrite the equations as \( \frac{x - 2}{3} = \cos \theta \) and \( \frac{y - 1}{3} = \sin \theta \).

Square both sides of each of these equations and add:

\[
\left( \frac{x - 2}{3} \right)^2 + \left( \frac{y - 1}{3} \right)^2 = \cos^2 \theta + \sin^2 \theta \quad \text{and therefore} \quad \frac{(x - 2)^2}{9} + \frac{(y - 1)^2}{9} = 1
\]

i.e. \( (x - 2)^2 + (y - 1)^2 = 9 \)

Ellipses

It has been shown in the previous section that \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), where \( a \) and \( b \) are positive real numbers, is the cartesian equation of an ellipse with centre at the origin, and \( x \)-axis intercepts \((\pm a, 0)\) and \( y \)-axis intercepts \((0, \pm b)\). The parametric equations for such an ellipse are \( x = a \cos t \) and \( y = b \sin t \). The minimal interval of \( t \) values to yield the entire ellipse is \([0, 2\pi]\).

The domain and range of the corresponding cartesian relation can be determined by the parametric equation determining the \( x \) value and the \( y \) value respectively. The range of the function with rule \( x = a \cos t \) is \([-a, a]\) and hence the domain of the relation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is \([-a, a]\). The range of the function with rule \( y = b \sin t \) is \([-b, b]\) and hence the range of the relation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is \([-b, b]\).

The proof that the two forms of equation yield the same graph uses the Pythagorean identity \( \sin^2 t + \cos^2 t = 1 \).

Let \( x = a \cos t \) and \( y = b \sin t \).

Therefore \( \frac{x}{a} = \cos t \) and \( \frac{y}{b} = \sin t \). Squaring both sides of each of these equations yields

\[
\frac{x^2}{a^2} = \cos^2 t \quad \text{and} \quad \frac{y^2}{b^2} = \sin^2 t
\]

Now, since \( \sin^2 t + \cos^2 t = 1 \) it follows that \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

Example 32

Find the cartesian equation of the curve with parametric equations \( x = 3 + 3 \sin t \), \( y = 2 - 2 \cos t \) with \( t \in \mathbb{R} \) and describe the graph.
Solution

\[ x = 3 + 3 \sin t \] and \[ y = 2 - 2 \cos t, \] therefore \[ \frac{x - 3}{3} = \sin t \] and \[ \frac{2 - y}{2} = \cos t. \]

Square both sides of each equation and add:

\[ \frac{(x - 3)^2}{9} + \frac{(2 - y)^2}{4} = \sin^2 t + \cos^2 t \]

Hence \[ \frac{(x - 3)^2}{9} + \frac{(2 - y)^2}{4} = 1 \]

But \( (2 - y)^2 = (y - 2)^2 \) so this equation is more neatly written as

\[ \frac{(x - 3)^2}{9} + \frac{(y - 2)^2}{4} = 1 \]

Clearly this is an ellipse, with centre at \((3, 2)\), and axes intercepts at \((3, 0)\) and \((0, 2)\).

Hyperbolas

The general cartesian equation for a hyperbola with ‘centre’ at the origin is \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \). The parametric equations are \( x = a \sec t \) and \( y = b \tan t \) where \( \sec t = \frac{1}{\cos t} \) and \( t \in \left( \frac{-\pi}{2}, \frac{\pi}{2} \right) \) gives the right-hand branch of the hyperbola. For the function with rule \( x = a \sec t \) and domain \( \left( \frac{-\pi}{2}, \frac{\pi}{2} \right) \) the range is \([a, \infty)\). (The sec function is discussed further in Chapter 3.)

The graph of \( y = a \sec x \) is shown for the interval \( \left( \frac{-\pi}{2}, \frac{\pi}{2} \right) \).

For the function with rule \( y = b \tan t \) and domain \( \left( \frac{-\pi}{2}, \frac{\pi}{2} \right) \) the range is \( R \). The left branch of the hyperbola can be obtained for \( t \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \).

The proof that the two forms of equation can yield the same graph uses a form of the Pythagorean identity \( \sin^2 t + \cos^2 t = 1 \). Divide both sides of this identity by \( \cos t \). This yields \( \tan^2 t + 1 = \sec^2 t \).

Consider \( x = a \sec t \) and \( y = b \tan t \). Therefore \( \frac{x}{a} = \sec t \) and \( \frac{y}{b} = \tan t \).

Square both sides of each equation to obtain \( \frac{x^2}{a^2} = \sec^2 t \) and \( \frac{y^2}{b^2} = \tan^2 t \).

Now, since \( \tan^2 t + 1 = \sec^2 t \), it follows that \( \frac{y^2}{b^2} + 1 = \frac{x^2}{a^2} \).

Therefore \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \).
**Example 33**

Find the cartesian equation of the curve with parametric equations \( x = 3 \sec t, \ y = 4 \tan t \), where \( t \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \) and describe the curve.

**Solution**

Now \( x = 3 \sec t \) and \( y = 4 \tan t \). Therefore \( \frac{x}{3} = \sec t \) and \( \frac{y}{4} = \tan t \). Square both sides of each equation to obtain \( \frac{x^2}{9} = \sec^2 t \) and \( \frac{y^2}{16} = \tan^2 t \). Add these two equations to obtain \( \frac{y^2}{16} + 1 = \frac{x^2}{9} \).

So the cartesian form of the curve is \( \frac{x^2}{9} - \frac{y^2}{16} = 1 \).

The range of the function with rule \( x = 3 \sec t \) for \( t \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \) is \((-\infty, -3]\). Hence the domain for the graph is \((-\infty, -3]\).

This is the left branch of a hyperbola, with centre at the origin, and \( x \) intercept at \((-3,0)\) and with asymptotes with equations \( y = \frac{4x}{3} \) and \( y = -\frac{4x}{3} \).

**Example 34**

Give parametric equations for each of the following:

\( a \) \( x^2 + y^2 = 9 \)  
\( b \) \( \frac{x^2}{16} + \frac{y^2}{4} = 1 \)  
\( c \) \( \frac{(x-1)^2}{9} - \frac{(y+1)^2}{4} = 1 \)

**Solution**

\( a \) The parametric equations are \( x = 3 \cos t \) and \( y = 3 \sin t \) or \( x = 3 \sin t \) and \( y = 3 \cos t \). There are infinitely many pairs of equations which determine the curve given by the cartesian equation \( x^2 + y^2 = 9 \). Others are \( x = -3 \cos(2t) \) and \( y = 3 \sin(2t) \). For \( x = 3 \cos t \) and \( y = 3 \sin t \) it is sufficient for \( t \) to be chosen for the interval \([0, 2\pi]\) to obtain the whole curve. For \( x = -3 \cos(2t) \) and \( y = 3 \sin(2t) \) it is sufficient for \( t \) to be chosen in the interval \([0, \pi]\).

\( b \) The obvious solution is \( x = 4 \cos t \) and \( y = 2 \sin t \)

\( c \) The obvious solution is \( x - 1 = 3 \sec t \) and \( y + 1 = 2 \tan t \)

**Using a graphics calculator with parametric equations**

Press [MODE] and select [Par] from the fourth row of the menu. Ensure that the calculator is in radian mode.
In the \( Y = \) screen enter \( X_{1T} = 2 \cos 3T \) and \( Y_{1T} = 2 \sin 3T \). The \( T \) is obtained by pressing the key \( \text{X}\text{T}\text{N} \).

The period of both functions is \( \frac{2\pi}{3} \). Press \( \text{WINDOW} \) and complete the settings as shown.

From the \( \text{ZOOM} \) menu select \( \text{Zsquare} \) and activate \( \text{TRACE} \). The \( \text{TRACE} \) can be used to see the order of plotting.

**Using a CAS calculator with parametric equations**

Press \( \text{MODE} \) and select \( \text{PARAMETRIC} \) as shown. Enter the functions in the \( Y = \) screen.

From \( \text{WINDOW} \) establish the settings as shown.

From the \( \text{ZOOM} \) menu choose \( \text{ZoomSquare} \). Activate \( \text{TRACE} \).

**Exercise**

1. Find the cartesian equation of the curve determined by the parametric equations \( x = 2 \cos 3t \) and \( y = 2 \sin 3t \), and determine the domain and range of the corresponding relation.
2 Determine the corresponding cartesian equation of the curve determined by each of the following parametric equations and sketch the graph of each of these.

a. \( x = \sec t, y = \tan t, t \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \)

b. \( x = 3 \cos 2t, y = -4 \sin 2t \)

c. \( x = 3 - 3 \cos t, y = 2 + 2 \sin t \)

d. \( x = 3 \sin t, y = 4 \cos t, t \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \)

e. \( x = \sec t, y = \tan t, t \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \)

f. \( x = 1 - \sec (2t), y = 1 + \tan(2t), t \in \left( \frac{\pi}{4}, \frac{3\pi}{4} \right) \)

3 Give parametric equations corresponding to each of the following:

a. \( x^2 + y^2 = 16 \)

b. \( \frac{x^2}{9} - \frac{y^2}{4} = 1 \)

c. \( (x - 1)^2 + (y + 2)^2 = 9 \)

d. \( \frac{(x - 1)^2}{9} + \frac{(y + 3)^2}{4} = 9 \)

4 A circle has centre \((1, 3)\) and radius 2. If parametric equations for this circle are \(x = a + b \cos(2\pi t)\) and \(y = c + d \sin(2\pi t)\), where \(a, b, c\) and \(d\) are positive constants, state the values of \(a, b, c\) and \(d\).

5 An ellipse has \(x\)-axis intercepts \((-4, 0)\) and \((4, 0)\) and \(y\)-axis intercepts \((0, 3)\) and \((0, -3)\). State a possible pair of parametric equations for this ellipse.

6 The graph of the circle with parametric equations \(x = 2 \cos 2t\) and \(y = 2 \sin 2t\) is dilated by a factor 3 from the \(x\) axis. For the image curve, state:

a. a possible pair of parametric equations

b. the cartesian equation

7 The graph of the ellipse with parametric equations \(x = 3 - 2 \cos \left( \frac{t}{2} \right)\) and \(y = 4 + 3 \sin \left( \frac{t}{2} \right)\) is translated 3 units in the negative direction of the \(x\) axis and 2 units in the negative direction of the \(y\) axis. For the image curve state:

a. a possible pair of parametric equations

b. the cartesian equation

8 Sketch the graph of the curve with parametric equations \(x = 2 + 3 \sin(2\pi t)\) and \(y = 4 + 2 \cos(2\pi t)\) for:

a. \( t \in \left[ 0, \frac{1}{3} \right] \)

b. \( t \in \left[ 0, \frac{1}{4} \right] \)

c. \( t \in \left[ 0, \frac{3}{4} \right] \)

For each of these, state the domain and range.
Summary of circles, ellipses and hyperbolas

**Circles**
- The circle with centre at the origin and radius \( a \) is the graph of the equation \( x^2 + y^2 = a^2 \).
- The circle with centre \((h, k)\) and radius \( a \) is the graph of the equation \((x - h)^2 + (y - k)^2 = a^2\).
- In general, \( x^2 + y^2 = a^2 \), where \( a > 0 \), is the cartesian equation of a circle with centre at the origin and radius \( a \). The parametric equations are \( x = a \cos t \) and \( y = a \sin t \). The minimal interval of \( t \) values to yield the entire circle is \([0, 2\pi]\).
- The circle with centre \((h, k)\) and radius \( a \) can be described through the parametric equations \( x = h + a \cos t \) and \( y = k + a \sin t \).

**Ellipses**
- The curve with equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is an ellipse with centre the origin, \( x \)-axis intercepts \((-a, 0)\) and \((a, 0)\) and \( y \)-axis intercepts \((0, -b)\) and \((0, b)\). For \( a > b \) the ellipse will appear as shown in the diagram on the left. If \( b > a \) the ellipse is as shown in the diagram on the right.

- The curve with equation \( \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \) is an ellipse with centre \((h, k)\).
- The ellipse with centre at the origin, and \( x \)-axis intercepts \((\pm a, 0)\) and \( y \)-axis intercepts \((0, \pm b)\) has parametric equations \( x = a \cos t \) and \( y = b \sin t \). The minimal interval of \( t \) values to yield the entire ellipse is \([0, 2\pi]\).
- The ellipse with centre \((h, k)\) formed by translating the ellipse with equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) can be described through the parametric equations \( x = h + a \cos t \) and \( y = k + b \sin t \).
Hyperbolas

- The curve with equation \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) is a hyperbola with centre at the origin. The axis intercepts are \((a, 0)\) and \((-a, 0)\). The hyperbola has asymptotes \( y = \frac{b}{a}x \) and \( y = -\frac{b}{a}x \).

- The curve with equation \( \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \) is a hyperbola with centre \((h, k)\). The hyperbola has asymptotes \( y - h = \frac{b}{a}(x - k) \) and \( y - h = -\frac{b}{a}(x - k) \).

- The parametric equations for the hyperbola shown above are \( x = a \sec t \) and \( y = b \tan t \) where \( \sec t = \frac{1}{\cos t} \).

- The hyperbola with centre \((h, k)\) formed by translating the hyperbola with equation \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) can be described through the parametric equations \( x = h + a \sec t \) and \( y = k + b \tan t \).

Multiple-choice questions

1. The 3rd term of a geometric sequence is 4. If the 8th term is 128, then the 1st term is:
   A 2  B 1  C 32  D 5  E none of these

2. If the numbers 5, \(x\) and \(y\) are in arithmetic sequence then:
   A \(y = x + 5\)  B \(y = x - 5\)  C \(y = 2x + 5\)
   D \(y = 2x - 5\)  E none of these

3. If \(2 \cos x^\circ - \sqrt{2} = 0\), the value of the acute angle \(x^\circ\) is:
   A 30°  B 60°  C 45°  D 25°  E 27.5°

4. The equation of the graph shown is:
   A \(y = \sin 2\left(x - \frac{\pi}{4}\right)\)
   B \(y = \cos \left(x + \frac{\pi}{4}\right)\)
   C \(y = \sin(2x)\)
   D \(y = -2 \sin(x)\)
   E \(y = \sin \left(x + \frac{\pi}{4}\right)\)

5. The exact value of the expression \(\sin \left(\frac{2\pi}{3}\right) \times \cos \left(\frac{\pi}{4}\right) \times \tan \left(\frac{\pi}{6}\right)\) is:
   A \(\frac{1}{\sqrt{2}}\)  B \(\frac{1}{\sqrt{3}}\)  C \(\frac{\sqrt{2}}{4}\)  D \(\frac{\sqrt{3}}{2}\)  E none of these
6 In the diagram, $A, B, C$ and $D$ are points on the circle. $\angle ABD = 35^\circ$ and $\angle AXB = 100^\circ$. The magnitude of $\angle XDC$ is:

- **A** $35^\circ$
- **B** $40^\circ$
- **C** $45^\circ$
- **D** $50^\circ$
- **E** $55^\circ$

7 In a geometric sequence $t_2 = 24$ and $t_4 = 54$. The sum of the first 5 terms, if the common ratio is positive, is:

- **A** $130$
- **B** $211$
- **C** $238$
- **D** $316.5$
- **E** $810$

8 In a triangle $ABC$, $a = 30$, $b = 21$ and $\cos C = \frac{21}{33}$. The value of $c$ to the nearest whole number is:

- **A** $9$
- **B** $10$
- **C** $11$
- **D** $81$
- **E** $129$

9 The coordinates of the centre of the circle with equation $x^2 - 8x + y^2 - 2y = 8$ are:

- **A** $(-8, -2)$
- **B** $(8, 2)$
- **C** $(-4, -1)$
- **D** $(4, 1)$
- **E** $(1, 4)$

10 The equation of the graph shown is:

- **A** $\frac{(x + 2)^2}{27} - \frac{y^2}{108} = 1$
- **B** $\frac{(x - 2)^2}{9} - \frac{y^2}{34} = 1$
- **C** $\frac{(x + 2)^2}{81} - \frac{y^2}{324} = 1$
- **D** $\frac{(x - 2)^2}{81} - \frac{y^2}{324} = 1$
- **E** $\frac{(x + 2)^2}{9} - \frac{y^2}{36} = 1$

**Short-answer questions (technology-free)**

1 For the difference equation $f_n = 5f_{n-1}, f_0 = 1$, find $f_n$ in terms of $n$.

2 $AP$ and $BP$ are tangents to the circle with centre $O$. If $AP = 10$ cm, find $OP$. 
3 Write down the equation of the ellipse shown.

\[
\frac{(x-2)^2}{9} + \frac{(y-7)^2}{49} = 1
\]

4 Find \( \sin \theta \).

5 Find \( x \).

6 A circle has a chord of length 10 cm situated 3 cm from its centre. Find:
   a the radius length
   b the angle subtended by the chord at the centre

7 a Find the exact value of \( \cos 315^\circ \).
   b Given that \( \tan x^\circ = \frac{3}{4} \) and \( 180^\circ < x < 270^\circ \), find an exact value of \( \cos x^\circ \).
   c Find an angle \( A (A \neq 330) \) such that \( \sin A = \sin 330^\circ \).

8 In the diagram, \( AD \) is a tangent to the circle with centre \( O \), \( AC \) intersects the circle at \( B \), and \( BD = AB \).
   a Find \( \angle BCD \) in terms of \( x \).
   b If \( AD = y \) cm, \( AB = a \) cm and \( BC = b \) cm, express \( y \) in terms of \( a \) and \( b \).

9 \( ABC \) is a horizontal right-angled triangle with the right angle at \( B \). \( P \) is a point 3 cm directly above \( B \). The length of \( AB \) is 1 cm and the length of \( BC \) is 1 cm.
   Find, the angle which the triangle \( ACP \) makes with the horizontal.

10 a Solve \( 2 \cos(2x + \pi) - 1 = 0, -\pi \leq x \leq \pi \).
    b Sketch the graph of \( y = 2 \cos(2x + \pi) - 1, -\pi \leq x \leq \pi \), clearly labelling axes intercepts.
    c Solve \( 2 \cos(2x + \pi) < 1, -\pi \leq x \leq \pi \).
11 The triangular base $ABC$ of a tetrahedron has side lengths $AB = 15$ cm, $BC = 12$ cm and $AC = 9$ cm. If the apex $D$ is 9 cm vertically above $C$, then find:
   a) the angle $C$ of the triangular base
   b) the angles that the sloping edges make with the horizontal

12 Two ships sail from port at the same time. One sails 24 nautical miles due east in three hours, and the other sails 33 nautical miles on a bearing of 030° in the same time.
   a) How far apart are the ships three hours after leaving port?
   b) How far apart would they be in five hours if they maintained the same bearings and constant speed?

13 Find $x$.

14 An airport $A$ is 480 km due east of airport $B$. A pilot flies on a bearing of 225° from $A$ to $C$ and then on a bearing of 315° from $C$ to $B$.
   a) Make a sketch of the situation.
   b) Determine how far the pilot flies from $A$ to $C$.
   c) Determine the total distance the pilot flies.

15 Find the equations of the asymptotes for the hyperbola with rule $x^2 - \frac{(y - 2)^2}{9} = 15$. 

16 A curve is defined by the parametric equations $x = 3 \cos(2t) + 4$ and $y = \sin(2t) - 6$. Give the cartesian equation of the curve.

17 a) Find the value of $x$.
   b) Find $a$, $b$, $c$ and $d$, given that $PR$ is a tangent to the circle with centre $O$.

18 A curve is defined by the parametric equations $x = 2 \cos(\pi t)$ and $y = 2 \sin(\pi t) + 2$. Give the cartesian equation of the curve.
19  a Sketch the graphs of \( y = -2 \cos x \) and \( y = -2 \cos\left(x - \frac{\pi}{4}\right) \) on the same set of axes, for \( x \in [0, 2\pi] \).
   b Solve \(-2 \cos\left(x - \frac{\pi}{4}\right) = 0\) for \( x \in [0, 2\pi] \).
   c Solve \(-2 \cos x \leq 0\) for \( x \in [0, 2\pi] \).

20  Find all angles \( \theta \), such that \( 0 \leq \theta \leq 2\pi \), where:
   a \( \sin \theta = \frac{1}{2} \)
   b \( \cos \theta = \frac{\sqrt{3}}{2} \)
   c \( \tan \theta = 1 \)

21  A circle has centre \((1, 2)\) and radius 3. If parametric equations for this circle are
   \( x = a + b \cos(2\pi t) \) and \( y = c + d \sin(2\pi t) \), where \( a, b, c \) and \( d \) are positive constants, state
   the values of \( a, b, c \) and \( d \).

22  \( O \) is the centre of a circle with points \( A, C, D \) and
   \( E \) on the circle. Find:
   a \( \angle ADB \)
   b \( \angle AEC \)
   c \( \angle DAC \)

23  Find the centre and radius of the circle with equation \( x^2 + 8x + y^2 - 12y + 3 = 0 \).

24  Find the \( x \)- and \( y \)-axes intercepts of the graph of the ellipse \( \frac{x^2}{81} + \frac{y^2}{9} = 1 \).

25  The first term of an arithmetic sequence is \((3p + 5)\) where \( p \) is a positive integer. The last
   term is \((17p + 17)\) and the common difference is 2.
   a Find in terms of \( p \):
      i the number of terms  ii the sum of the sequence
   b Show that the sum of the sequence is divisible by 14 only when \( p \) is odd.

26  A sequence is formed by using rising powers of 3: \( 3, 3^1, 3^2, \ldots \)
   a Find the \( n \)th term.  b Find the product of the first twenty terms.

Extended-response questions

1  A hiker walks from point \( A \) on a bearing of \( 010^\circ \) for 5 km and then on a bearing of \( 075^\circ \) for
   7 km to reach point \( B \).
   a Find the length of \( AB \).
   b Find the bearing of \( B \) from the start point \( A \).
   A second hiker travels from point \( A \) on a bearing of \( 080^\circ \) for 4 km to a point \( P \), and then
   travels in a straight line to \( B \).
   c Find:
      i the total distance travelled by the second hiker
      ii the bearing on which the hiker must travel in order to reach \( B \) from \( P \).
A third hiker also travels from point $A$ on a bearing of $080^\circ$ and continues on that bearing until he reaches point $C$. He then turns and walks towards $B$. In doing so, the two legs of the journey are of equal length.

d Find the distance travelled by the third hiker to reach $B$.

2 An ellipse is defined by the rule $\frac{x^2}{2} + \frac{(y + 3)^2}{5} = 1$.

a Find:

i the domain of the relation

ii the range of the relation

iii the centre of the ellipse.

$E$ is an ellipse given by the rule $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$. The domain of $E$ is $[-1, 3]$ and its range is $[-1, 5]$.

b Find the values of $a$, $b$, $h$ and $k$.

The line $y = x - 2$ intersects the ellipse $E$ at $A (1, -1)$ and at $P$.

c Find the coordinates of the point $P$.

A line perpendicular to the line $y = x - 2$ is drawn at $P$. This line intersects the $y$ axis at $Q$.

d Find the coordinates of $Q$.

e Find the equation of the circle through $A$, $P$ and $Q$.

3 a Show that the circle with equation $x^2 + y^2 - 2ax - 2ay + a^2 = 0$, touches both the $x$ axis and the $y$ axis.

b Show that every circle that touches the $x$ axis and $y$ axis has an equation of a similar form.

c Hence show that there are exactly two circles passing through the point $(2, 4)$ and just touching the $x$ axis and $y$ axis and give their equations.

d State the coordinates of the centres of these two circles and give the radius of each of these circles.

e For each of the circles, find the gradient of the line which passes through the centre and the point $(2, 4)$.

f Find an equation to the tangent to each circle at the point $(2, 4)$.

4 A circle is defined by the parametric equation $x = a \cos \theta$ and $y = a \sin \theta$. Let $P$ be the point with coordinates $(a \cos \theta, a \sin \theta)$.

a Find the equation of the straight line which passes through the origin and the point $P$.

b State the coordinates, in terms of $\theta$, of the other point of intersection of the circle with the straight line through the origin and $P$.

c Find the equation of the tangent to the circle at the point $P$.

d Find the coordinates of the points of intersection $A$ and $B$ of the tangent with the $x$ axis and $y$ axis respectively.

e Find the area of triangle of $OAB$ in terms of $\theta$ if $0 < \theta < \frac{\pi}{2}$. Find the value of $\theta$ for which the area of this triangle is a minimum.
5 The line with equation \( x = -a \) is the equation of the side \( BC \) of an equilateral triangle \( ABC \) circumscribing the circle with equation \( x^2 + y^2 = a^2 \).
   a Find the equations of \( AB \) and \( AC \).
   b Find the equation of the circle circumscribing triangle \( ABC \).

6 This diagram shows a straight track through points \( A, S \) and \( B \), where \( A \) is 10 km northwest of \( B \) and \( S \) is exactly halfway between \( A \) and \( B \). A surveyor is required to reroute the track through \( P \) from \( A \) to \( B \) to avoid a major subsidence at \( S \).
   The surveyor determines that \( A \) is on a bearing of \( 330^\circ \) from \( P \) and \( B \) is on a bearing of \( 70^\circ \) from \( P \).
   Assume that the region under consideration is flat. Find:
   a the magnitude of angles \( APB, PAB \) and \( PBA \)
   b the distance from \( P \) to \( B \) and from \( P \) to \( S \)
   c the bearing of \( S \) from \( P \)
   d the distance from \( A \) to \( B \) through \( P \), if the surveyor chooses to reroute the track along a circular arc.